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An exactly solvable class of discrete Schrödinger equations

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Abstract. A class of potentials admitting analytic solution of the discrete Schrödinger equation in terms of q-hypergeometric functions is considered. In particular, it includes the exponential and Hulthén potentials as well as the Maryland model.

Exactly solvable models related to discrete Schrödinger operator on $l^2(Z)$

$$(H\Psi)(x) = \Psi(x+1) + \Psi(x-1) + V(x)\Psi(x) \tag{1}$$

are of much interest in solid state physics and statistical physics (see reviews [1, 2] and references therein). The short list of such models known up to now consists of the linear [3], quadratic [4] and Coulomb [5, 6] potentials plus the so-called Maryland model [7, 8] with an almost periodic potential

$$V(x) = \lambda \tan(\pi\alpha x + \theta). \tag{2}$$

In this paper we present a new class of potentials for which the discrete Schrödinger equation admits explicit solution:

$$V(x) = \omega(q^x) \tag{3}$$

where $q \in \mathbb{C}$ is a parameter and $\omega(z)$ is of the form

$$\omega(z) = \frac{a+bz}{1+cz}. \tag{4}$$

In particular, this class includes the Maryland model (2) that corresponds to the following choice of parameters:

$$q = \mathrm{e}^{-2\pi\mathrm{i}\alpha} \qquad a = -\mathrm{i}\lambda \qquad b = \mathrm{i}\lambda\mathrm{e}^{-2\mathrm{i}\theta} \qquad c = \mathrm{e}^{-2\mathrm{i}\theta} \tag{5}$$

as well as two other interesting examples: exponential potential

$$V(x) = \lambda\mathrm{e}^{-\alpha|x|} \tag{6}$$

$$q = \mathrm{e}^{-\alpha} \qquad a = c = 0 \qquad b = \lambda$$

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and the Hulthén potential

$$V(x) = \lambda \frac{e^{-\alpha|x|}}{(1 - e^{-\alpha|x|})} \tag{7}$$

$$q = e^{-\alpha} \qquad a = 0 \qquad b = \lambda \qquad c = -1.$$

We show that the eigenfunctions of operator (1) with such a potential are expressed in terms of the q-hypergeometric function. To this end, let us work with the new variable $z = q^x$ in terms of which the discrete Schrödinger equation with potential (3) becomes a q-difference equation

$$\Psi(qz) + \Psi(z/q) + \left(\frac{a + bz}{1 + cz} - E\right)\Psi(z) = 0. \tag{8}$$

We look for its solution in the form

$$\Psi(z) = \sum_{n=0}^{\infty} f_n z^{n+\sigma}. \tag{9}$$

Substituting this into (8) and equating powers of z reduces the three-term recurrence equation (8) to a two-term recurrence relation for the coefficients of the ansatz (9)

$$(q^{n+\sigma} + q^{-n-\sigma} - a - E) = (b - c[q^{n-1+\sigma} + q^{-n+1-\sigma} - E])f_{n-1} \tag{10}$$

where σ solves the equation

$$q^{\sigma} + q^{-\sigma} = a + E. \tag{11}$$

Two solutions to this equation ($\pm\sigma$) fix two independent solutions to equation (8).

By making use of the identity

$$q^{\nu} + q^{-\nu} - A = q^{-\nu}(1 - q^{\nu+\tau})(1 - q^{\nu-\tau})$$

where

$$q^{\tau} + q^{-\tau} = A$$

equation (10) can be solved in terms of the q-shifted factorials

$$(a; q)_n = \begin{cases} 1 & n = 0 \\ (1-a)(1-aq)\cdots(1-aq^{n-1}) & n = 1, 2, \ldots . \end{cases}$$

Namely, if the parameter c of the potential (4) is zero, the solution to (10) is

$$f_n = \frac{b^n q^{n\sigma} q^{n(n+1)/2}}{(q; q)_n (q^{2\sigma+1}; q)_n} f_0.$$

For $c \neq 0$ one gets

$$f_n = \frac{(-qc)^n \; (q^{\sigma+\tau}; q)_n \; (q^{\sigma-\tau}; q)_n}{(q; q)_n \; (q^{2\sigma+1}; q)_n} f_0$$

where σ is defined by (11) and τ solves

$$q^{\tau} + q^{-\tau} = E + \frac{b}{c}.$$

By making use of these expressions series (9) can be converted into q-hypergeometric function ${}_r\varphi_s$ [9]:

(i) $c = 0$:

$$\Psi(x) = q^{\sigma x}\, {}_1\varphi_1(0; q^{2\sigma+1}; q; -bq^{x+\sigma+1}) \tag{12}$$

(ii) $c \neq 0$:

$$\Psi(x) = q^{\sigma x}\, {}_2\varphi_1(q^{\sigma+\tau} q^{\sigma-\tau}; q^{2\sigma+1}; q; -cq^{x+1}). \tag{13}$$

These formulae yield explicit expressions for the solutions to the discrete Schrödinger equation with potential (3). Note that equation (12) can be rewritten in terms of the Hahn–Exton q-Bessel function $J_{\nu}(z; q)$:

$$\Psi(x) = J_{2\sigma}(\sqrt{-b}\, q^{(x+\sigma)/2}; q) \qquad (c = 0). \tag{14}$$

We omit constant normalization factors in equations (12)–(14).

Let us now turn to particular examples. Consider, for instance, the even eigenstates ($\Psi(x) = \Psi(-x)$) of the operator (1) which are fixed by the boundary condition $\Psi(0) = 0$. Then in the case of the exponential potential (6) the corresponding eigenvalues E_i are given by the transcendental equation

$$J_{2\sigma(E_i)}\left(\left[-\lambda \mathrm{e}^{-\alpha\sigma(E_i)}\right]^{1/2}; \mathrm{e}^{-\alpha}\right) = 0 \tag{15}$$

where σ is the positive root of equation (11) (to provide exponential decay of the eienfunctions as $x \to \infty$):

$$\sigma(E) = \alpha^{-1} \ln\left(\tfrac{1}{2}\left[E + \sqrt{E^2 - 4}\right]\right). \tag{16}$$

For the Hulthén potential (7) an analogous equation is solved explicitly by making use of Heine's q-analogue of Gauss' summation formula

$$\Psi(0) = {}_2\varphi_1\left(q^{\sigma+\tau}, q^{\sigma-\tau}; q^{2\sigma+1}; q; q\right) = \frac{(q^{\sigma-\tau+1}; q)_{\infty}(q^{\sigma+\tau+1}; q)_{\infty}}{(q^{2\sigma+1}; q)_{\infty}(q; q)_{\infty}} \tag{17}$$

where $q = \mathrm{e}^{-\alpha}$, $\sigma(E)$ is given by (16) and

$$\tau(E) = \alpha^{-1} \ln\left(\tfrac{1}{2}\left[E - \lambda + \sqrt{(E-\lambda)^2 - 4}\right]\right). \tag{18}$$

Expression (17) vanishes when

$$\sigma(E) + \tau(E) = -n \qquad n = 1, 2, \ldots \tag{19}$$

that gives the quantization rule for the eigenvalues.

The scattering problem related to these two models is also solved exactly. The eigenfunctions of the continuous spectrum $E = 2\cos k$ ($k \in (0, \pi)$) are linear combinations of the solutions (12) or (13) with different signs of $\sigma, \sigma = \pm \mathrm{i}k/\alpha$. In the limit $x \to \infty$ they are of the form

$$\Psi(x, k) \sim \sin(kx + \delta(k)).$$

The phase shift is determined by condition at $x = 0$. For instance, for the even states one gets

(i) the exponential potential:

$$\delta(k) = \arg\{J_{2\mathrm{i}k/\alpha}(\sqrt{-\lambda}\mathrm{e}^{-\mathrm{i}k/2}; \mathrm{e}^{-\alpha})\} \tag{20}$$

(ii) the Hulthén potential:

$$\delta(k) = \arg\left\{\frac{(q^{\sigma-\tau+1}; q)_\infty (q^{\sigma+\tau+1}; q)_\infty}{(q^{2\sigma+1}; q)_\infty}\right\} \tag{21}$$

where $\sigma = \mathrm{i}k/\alpha$ and τ is defined by (18). Note that quantization rules (15), (19) and the expressions (20), (21) for the phase shifts are q-generalizations of the well known results for the continuous Schrödinger equation with exponential or Hulthén potentials [10].

For the Maryland model (2), (5) formula (13) yields a simple representation of eigenfunctions that seems to be previously unknown. Note that the arguments of the q-function on the right-hand side of equation (13) in this case are on the unit circle which is the boundary of the convergence domain of the corresponding basic hypergeometric series. In such a case the latter is to be understood as analytical continuation provided, for instance, by the q-analogue of Barnes' integral [9].

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